Int J Theor Phys (2010) 49: 365–375
DOI 10.1007/s10773-009-0209-5

Hamiltonian Structure of Fractional First Order Lagrangian

Ali Khalili Golmankhaneh · Alireza Khalili Golmankhaneh · Dumitru Baleanu · Mihaela Cristina Baleanu

Received: 19 August 2009 / Accepted: 25 November 2009 / Published online: 9 December 2009

Abstract In this paper, we show that the fractional constraint Hamiltonian formulation, using Dirac brackets, leads to the same equations as those obtained from fractional Euler-Lagrange equations. Furthermore, the fractional Faddeev-Jackiw formalism was constructed.

Keywords Fractional dynamics · Fractional derivatives · Fractional Faddeev-Jackiw · Constraint Hamiltonian · Dirac brackets

1 Introduction

Derivatives and integrals of fractional order have found many applications in recent studies in mechanics, physics and engineering. For example, in chaotic dynamics, quantum me-

A.K. Golmankhaneh · A.K. Golmankhaneh
Department of Physics, Islamic Azad University-Uromia Branch, P.O. Box 969, Uromia, Iran

A.K. Golmankhaneh
e-mail: ali.khalili.gol@gmail.com

A.K. Golmankhaneh
Department of Physics, University of Pune, Pune 411007, India
e-mail: alireza@physics.unipune.ernet.in

D. Baleanu (✉)
Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
e-mail: dumitru@cankaya.edu.tr

D. Baleanu
Institute of Space Sciences, P.O. Box MG-23, 76900 Magurele-Bucharest, Romania

M.C. Baleanu
Faculty of Physics, University of Bucharest, Bucharest, Romania

M.C. Baleanu
National Mihail Sadoveanu High School, District 2, Bucharest, Romania

chanics, plasma physics, anomalous diffusion, and so many fields of physics [14, 17, 18, 24, 28, 29, 36, 40, 42]. Especially in mechanics Riewe has shown that Lagrangian involving fractional time derivatives leads to equation of motion with non conservative classical derivatives such as friction [30, 31]. Motivated by this approach many researchers have explored this area giving new insight into this problem [2, 6, 7, 11, 15, 20, 21, 23, 26, 32–34, 38, 39, 41]. Agrawal has presented fractional Euler-Lagrange equation in Riemann-Liouville derivatives [1]. Fractional multi time Lagrangian equations for dynamical systems within Riemann-Liouville derivatives and fractional multi time Hamiltonian has introduced. The constant of dynamics of fractional multi time Hamilton equations are discussed [8]. The Newtonian equation with memory as physical model possessing memory effect for the application of fractional derivatives has been introduced [9, 10]. The fractional relativistic scalar fields for non conservative systems within the fractional Caputo derivatives has been derived [13]. Using illustrative example, we have explained the formalism.

Constrained systems are very important in science and engineering [16, 37]. The fractional single time Hamiltonian formulation has been developed in [5, 32].

The extension of fractional Hamiltonian formulation has been studied for fractional constrained systems [25, 26]. Fractional Hamiltonian systems with linearly dependent constraints has worked out involving fractional Riemann-Liouville derivatives [3]. Lagrangians linear in velocities has been studied using the fractional calculus [4, 27]. Wentzel-Kramer-Brillouin (WKB) approximation for fractional systems was investigated using the fractional calculus in [35]. Generalizing the techniques encountered in the classical dynamics of constrained systems is an open issue in the area of fractional calculus.

Our main purpose is to introduce the systematic treatment of constraint Hamiltonian system developed by Dirac [19] to fractional mechanics. We also apply method of Faddeev-Jackiw (F-J) to fractional Hamiltonian associated to first order Lagrangian [22].

This manuscript is organized as follows: Sect. 2 is devoted a brief review of the Fractional derivative definitions. In Sect. 3 we discuss the classical Hamiltonian formulation. In Sect. 4 we explain first order fractional Lagrangian using Dirac method by an example. Section 5 deals with the classical F-J formulation. In Sect. 6 we use F-J method, for the first order fractional Lagrangian. In Sect. 7 is a short conclusion.

2 Fractional Derivatives

In this section we briefly present the definition of the left and the right fractional derivatives of Riemann-Liouville as well as Caputo [14, 17, 24, 28, 29, 36, 40]. The left Riemann-Liouville fractional derivative is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{1} $$

and the right Riemann-Liouville fractional derivative,

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau, \tag{2} $$

where the order α fulfills $n-1 \le \alpha < n$ and Γ represent the gamma function. An alternative definition of Riemann-Liouville fractional derivative called Caputo derivative that introduced by Caputo in 1967. The left Caputo derivative defined as

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{3} $$

and the right Caputo fractional derivative

$$ {}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{4} $$

where the order α satisfies $n-1 \leq \alpha < n$. The Riemann-Liouville derivative of constant isn't zero, although Caputo derivative of a constant is zero.

3 Classical Hamiltonian Formulation

Our formal discussion of constrained systems begins with the consideration of functional action as

$$ S = \int_a^b L(q_i, \dot{q}_i) dt, \tag{5} $$

where q_i are a canonical coordinate and $\dot{q}_i$ are a canonical velocity. We confine ourself to Lagrangian without explicit time dependence, then the Euler-Lagrange equations is

$$ \frac{\partial L}{\partial q_i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \tag{6} $$

by requiring the variation of the action S to be stationary. If we choose for our Poisson bracket the convention

$$ \{u, v\}_P = \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p^i} - \frac{\partial u}{\partial p^i} \frac{\partial v}{\partial q_i}, \tag{7} $$

(indices of i obey the Einstein summation rule). We have

$$ \{p^i, q_i\} = -\delta^i_j, \tag{8} $$

where δ^i_j is Kronecker delta. Then the canonical Hamilton,

$$ H_c(p^i, q_i) = p^i q_i - L(q_i, \dot{q}_i), \tag{9} $$

formally generate the Hamilton equations of motion

$$ \dot{q}_i = \{q_i, H_c\} = +\frac{\partial H_c}{\partial p^i}, \tag{10} $$

$$ \dot{p}^i = \{p^i, H_c\} = -\frac{\partial H_c}{\partial q_i}, \tag{11} $$

where the index c denotes the canonically of H. A necessary and sufficient condition that L be singular is

$$ \mathrm{Det} \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = 0. \tag{12} $$

This is a sign that there exist certain *primary constraints*

$$ \phi_a(q, p) \approx 0, \tag{13} $$

following from the form of the Lagrangian alone. In order to have a consistent system, we require the time derivative of the constraints (13) to be zero

$$\dot{\phi}_a = \{\phi_a, H_T\} = \{\phi_a, H_c\} + u_m\{\phi_a, \phi_b\} \approx 0. \tag{14}$$

First, we may find that this equation gives no new information, but simply imposes conditions on the form of u_m. Second it may impose new relation among the p's and q's independent of u_m. These are *secondary constraints* and must be adjoined to the original constraints (13).

4 Fractional Constraint Hamiltonian Formulation

Let us consider a fractional Lagrange system [1]

$$L(q_i, {}_aD_t^\alpha q_i, t), \quad i = 1, 2, \ldots, N. \tag{15}$$

The dynamics of a fractional system is then determined by the condition that its action,

$$S = \int L(q_i, {}_aD_t^\alpha q_i, t)dt, \tag{16}$$

is stationary along trajectory $q^i(t)$. From the calculus of variations this leads to fractional Euler-Lagrange equations,

$$\frac{\partial L}{\partial q_i} + {}_bD_t^\alpha \frac{\partial L}{\partial_a D_t^\alpha} q_i = 0. \tag{17}$$

The fractional canonical Hamiltonian given by

$$H_c = p^i_{\alpha a} D_t^\alpha q_i - L. \tag{18}$$

This canonical Hamiltonian generates the fractional Hamilton equations [32]

$$\frac{\partial H_c}{\partial p^i} = {}_aD_t^\alpha q_i, \qquad \frac{\partial H_c}{\partial q_i} = {}_tD_b^\alpha p^i. \tag{19}$$

We choose for Poisson bracket the convention,

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p^i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p^i}. \tag{20}$$

Then we have

$$\{q_i, H\} = \frac{\partial q_i}{\partial q_i}\frac{\partial H}{\partial p^i} - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial p^i}, \tag{21}$$

and fractional Hamilton equations in terms of Poisson bracket

$${}_aD_t^\alpha q_i = \{q_i, H\} = \frac{\partial H}{\partial p^i}, \tag{22}$$

and in similar manner

$$-{}_tD_b^\alpha p^i = \{p^i, H\} = -\frac{\partial H}{\partial q_i}. \tag{23}$$

Note that in classical mechanics, $\dot{F} = \{F, H\}$, but in fractional one,

$$D_t^\alpha F \neq \{F, H_c\}. \tag{24}$$

Since

$$\{f, H\} = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p^i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p^i}, \tag{25}$$

and using the fractional Hamilton equations we lead

$$\{f, H\} = \frac{\partial f}{\partial p^i}{}_t D_b^\alpha p^i - \frac{\partial f}{\partial q_i}{}_a D_t^\alpha q_i. \tag{26}$$

Equation (26) shows the validity of (24). The Lagrangian linear in the velocities is a typical example of constrained system. In this case the Euler-Lagrange equations are not consistent with fractional Hamiltonian one. However, if we introduce Lagrange multipliers to the canonical Hamiltonian, then one can obtain the total Hamiltonian in the Dirac's formulation which is consistent with Euler-Lagrange equations of motion. In the next subsection we apply the classical Hamiltonian formulation for the fractional one to remove this ambiguity. When the Poisson bracket becomes zero, its concept is different from classical one.

4.1 A First Order Lagrangian

Consider a Lagrangian depending on the fractional time derivative of coordinate in the form $L(q, {}_a D_t^\alpha q, t)$. If the Lagrangian is linear in velocity the canonical Hamiltonian, (18), won't be unique. In this case the Euler-Lagrange equations (17) aren't consistent with fractional Hamilton equations (19). In the following examples we show that the classical Hamiltonian formulation could apply to fractional mechanics and the results are consistent.

Example 1 Consider the Lagrangian which are not quadratic in the velocity,

$$L(x, y, {}_a D_t^\alpha x, y, {}_a D_t^\alpha y) = x\, {}_a D_t^\alpha y - y\, {}_a D_t^\alpha x - V(x, y), \tag{27}$$

then the fractional Euler-Lagrange equations remain valid and give us the following equations of motion, x component,

$$\frac{\partial L}{\partial x} + {}_t D_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha x} = 0, \tag{28}$$

and

$$-\frac{\partial V}{\partial x} + {}_a D_t^\alpha y - {}_t D_b^\alpha y = 0, \tag{29}$$

and y component,

$$\frac{\partial L}{\partial y} + {}_t D_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha y} = 0, \tag{30}$$

$$-\frac{\partial V}{\partial y} - {}_a D_t^\alpha x + {}_t D_b^\alpha x = 0. \tag{31}$$

Let us define conjugate moments,

$$p_x = \frac{\partial L}{\partial_a D_t^\alpha x} = -y, \tag{32}$$

$$p_y = \frac{\partial L}{\partial_a D_t^\alpha y} = x. \tag{33}$$

We obtain the Hamiltonian

$$H = p_{x\,a} D_t^\alpha x + p_{y\,a} D_t^\alpha y - L = V(x, y). \tag{34}$$

Hamilton equations, p components,

$$\frac{\partial H}{\partial p_x} = {}_a D_t^\alpha x, \qquad \frac{\partial H}{\partial p_y} = {}_a D_t^\alpha y. \tag{35}$$

Then we have

$$0 = {}_a D_t^\alpha x, \qquad 0 = {}_a D_t^\alpha y. \tag{36}$$

The x, y components,

$$\frac{\partial H}{\partial x} = {}_t D_b^\alpha p_x, \qquad \frac{\partial H}{\partial y} = {}_t D_b^\alpha p_y, \tag{37}$$

then we have

$$\frac{\partial V}{\partial x} = {}_t D_b^\alpha p_x, \qquad \frac{\partial V}{\partial y} = {}_t D_b^\alpha p_y. \tag{38}$$

Equations (36) and (38) are not consistent with (29) and (31). At least the one is especial case of another. Using the method of the classical Hamiltonian formulation (Dirac) we could lead to fractional Hamilton equations which they are consistent with equations of fractional Euler Lagrangian. Now we generalize our Hamiltonian to

$$H_T = V(x, y) + u_1(y + p_x) + u_2(x - p_y), \tag{39}$$

where H_T is a total Hamiltonian and then apply fractional consistency condition

$$\{\phi_1, H\}_p = -\left(\frac{\partial V}{\partial x} + 2u_2\right) = 0, \tag{40}$$

and

$$\{\phi_2, H\}_p = \left(\frac{\partial V}{\partial y} + 2u_1\right) = 0. \tag{41}$$

These are not secondary conditions but consistency conditions for our arbitrary coefficients u_1 and u_2. But the ϕ_1 and ϕ_2 are second class constraints since

$$\{\phi_1, \phi_2\}_p = 2. \tag{42}$$

Then Fractional motion using the Dirac brackets, x component

$${}_a D_t^\alpha x = \{x, H\}_D = -\frac{1}{2}\frac{\partial V}{\partial y}, \tag{43}$$

y component

$$ {}_aD_t^\alpha y = \{y, H\}_D = \frac{1}{2}\frac{\partial V}{\partial x}, \tag{44} $$

and for momenta

$$ -{}_tD_b^\alpha p_x = \{p_x, H\}_D = -\frac{1}{2}\frac{\partial V}{\partial x}, \tag{45} $$

$$ -{}_tD_b^\alpha p_y = \{p_x, H\}_D = -\frac{1}{2}\frac{\partial V}{\partial y}. \tag{46} $$

Using $p_x = -y$ and $-\frac{1}{2}\frac{\partial V}{\partial x} = \frac{1}{2}\frac{\partial V}{\partial x} - \frac{\partial V}{\partial x}$ we lead to fractional Euler-Lagrange equations.

Example 2 Consider the linear fractional Lagrangian

$$ L = a(x)_a D_t^\alpha x + V(x), \tag{47} $$

and then the fractional Euler-Lagrange equations is

$$ \frac{\partial a}{\partial x}{}_aD_t^\alpha x + {}_tD_b^\alpha a - \frac{\partial V}{\partial x} = 0. \tag{48} $$

Now we define the canonical Hamiltonian

$$ H_c = {}_aD_t^\alpha x \frac{\partial L}{\partial_a D_t^\alpha x} - L, \tag{49} $$

and conjugate momenta

$$ p_x = \frac{\partial L}{\partial_a D_t^\alpha x} = a(x). \tag{50} $$

Therefore, the primary constant

$$ \phi_1 = a(x) - p_x, \tag{51} $$

and the effective or total Hamiltonian

$$ H_T = V(x) + u(a(x) - p_x) \tag{52} $$

generate the fractional Hamilton equations that are as follows:

$$ {}_aD_t^\alpha x = \{x, H_T\}_D = u \tag{53} $$

and

$$ -{}_tD_b^\alpha p_x = \{p_x, H_T\}_D = \{p_x, V(x) + u(p_x - a(x))\}_P, \tag{54} $$

which is consistent with fractional Euler equation (48).

5 The Classical Faddeev-Jackiw

In the Faddeev-Jackiw [12] procedure one starts from a Lagrange which is first order in time derivative and one writes the Lagrange density

$$L(q^i, \dot{q}^i, p_i, \dot{p}_i) = \begin{pmatrix} p_i & q^i \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} - H(p_i, q^i). \tag{55}$$

If we integrate L respect to time and apply the principle of stationary action we lead to Euler-Lagrange equations

$$S = \int L dt, \tag{56}$$

where

$$L(p_i, \dot{p}_i, q^i, \dot{q}^i) = \frac{1}{2}\{q^i \dot{p}_i - p\dot{q}^i\} - H(p_i, q^i). \tag{57}$$

Applying the principle least action we lead to Euler-Lagrange equation

$$\frac{\partial L}{\partial p_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i} = 0, \tag{58}$$

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{59}$$

If we replace the Lagrangian equation (57) we obtain the Hamilton equations. If we generalize the Lagrange

$$L = c_n(\xi)\dot{\xi}^n - V(\xi)dt, \tag{60}$$

where, $n = 1, \ldots, 2N$ in such way that

$$\xi^i = q^i, \qquad \xi^{N+i} = p_i. \tag{61}$$

Using least action principles, Euler-Lagrange equations

$$f_{nm}\dot{\xi}^m = \frac{\partial V}{\partial \xi^n}, \tag{62}$$

$$\dot{\xi}^n = f_{nm}^{-1}\frac{\partial V}{\partial \xi^m}. \tag{63}$$

If the brackets satisfies the relation

$$\{A, B\} = \frac{\partial A}{\partial \xi^n}\{\xi^n, \xi^m\}\frac{\partial B}{\partial \xi^m}, \tag{64}$$

$$\dot{\xi}^n = \{V, \xi^n\} = \frac{\partial V}{\partial \xi^m}\{\xi^m, \xi^n\}_{FJ}, \tag{65}$$

where

$$\{\xi^n, \xi^m\}_{FJ} = f_{nm}^{-1}. \tag{66}$$

For the simple case (57) the bracket (66) reduces to the usual

$$\{p_i, q^j\} = \delta_{ij}, \qquad \{q^i . q^j\} = \{p_i, p_j\} = 0, \tag{67}$$

or in other words

$$\{\xi^n, \xi^n\} = \epsilon^{nm} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{68}$$

This matrix is antisymmetric matrix.

6 Fractional Faddeev-Jackiw

In fractional mechanics consider the Lagrangian,

$$L = -\frac{1}{2}(q^i {}_t D_b^\alpha p_i + p_i {}_a D_t^\alpha q^i) + H(q^i, p_i). \tag{69}$$

In matrix form we have

$$L = \begin{pmatrix} \xi_n & \xi_m \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} {}_a D_t^\alpha \xi_n \\ {}_t D_b^\alpha \xi_m \end{pmatrix} - H(q, p). \tag{70}$$

Euler-Lagrange equations of this Lagrangian are

$$\frac{\partial L}{\partial q^i} + {}_t D_b^\alpha \frac{\partial L}{\partial_a D_t^\alpha q^i} = 0, \tag{71}$$

$$\frac{\partial L}{\partial p_i} + {}_a D_t^\alpha \frac{\partial L}{\partial_t D_b^\alpha p_i} = 0. \tag{72}$$

These equations are equivalence of fractional Hamilton equations. They are written in matrix form as follows

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} {}_a D_t^\alpha \xi_n \\ {}_t D_b^\alpha \xi_m \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial \xi^n} \\ \frac{\partial H}{\partial \xi^m} \end{pmatrix} \tag{73}$$

and we have

$$\begin{pmatrix} {}_a D_t^\alpha \xi_n \\ {}_t D_b^\alpha \xi_m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \xi^n} \\ \frac{\partial H}{\partial \xi^m} \end{pmatrix}. \tag{74}$$

In above relation we have used

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the basic Faddeev-Jackiw brackets are

$$\{q^i, p_j\}_{FJ} = \delta_{ij}, \qquad \{q^i, q^j\}_{FJ} = \{p_i, p_j\}_{FJ} = 0. \tag{75}$$

In bracket notation we have

$$\begin{pmatrix} {}_a D_t^\alpha \xi_n \\ {}_t D_b^\alpha \xi_m \end{pmatrix} = \{\xi^m, \xi^n\} \begin{pmatrix} \frac{\partial H}{\partial \xi^n} \\ \frac{\partial H}{\partial \xi^m} \end{pmatrix}, \tag{76}$$

where

$$\{\xi^m, \xi^n\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{77}$$

This matrix is symmetric contrast to classical one which it is anti symmetric.

7 Conclusion

Fractional mechanics describes both conservative and non conservative systems. Recently the fractional variation gained importance in the study of fractional mechanics. The fractional Lagrange and Hamiltonian formulations for the constraint systems are still at the beginning of their development. In this paper we consider a fractional first order Lagrangian formulation and we obtain the fractional motion equations using common Hamiltonian and Euler-Lagrange equations. We have observed that they are not the same. Then we use the classical constraint Hamiltonian formulation that lead us to the same equations that we obtain from Euler-Lagrange equations. Although we have used this formulation, the concept of classical brackets changes to a fractional concept. The Faddeev-Jackiw method in classical mechanics is another important method for quantization. In this manuscript we have developed its fractional counterpart.

References

1. Agrawal, O.P.: Formulation of Euler-Lagrange equations for fractional variational problems. J. Math. Anal. Appl. **272**, 368–379 (2002)
2. Atanackovic, T.M., Konjik, S., Pilipovic, S., et al.: Variational problems with fractional derivatives: Invariance conditions and Nother's theorem. Nonlinear Anal. **71**, 1504–1517 (2009)
3. Baleanu, D.: Fractional Hamiltonian analysis of irregular systems. Signal Process. **86**, 2632–2636 (2006)
4. Baleanu, D., Avkar, T.: Lagrangians with linear velocities within Riemann-Liouville fractional derivatives. Il Nuovo Cimento B **119**, 73–79 (2004)
5. Baleanu, D., Agrawal, O.P.: Fractional Hamilton formalism within Caputo's derivative. Czech. J. Phys. **56**, 1087–1092 (2006)
6. Baleanu, D., Muslih, S.I.: Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives. Phys. Scr. **72**, 119–121 (2005)
7. Baleanu, D., Muslih, S., Tas, K.: Fractional Hamiltonian analysis of higher order derivatives systems. J. Math. Phys. **47**, 103503 (2006)
8. Baleanu, D., Golmankhaneh, K.A., Golmankhaneh, K.A.: The dual action of fractional multi time Hamilton equations. Int. J. Theor. Phys. **48**, 2558–2569 (2009)
9. Baleanu, D., Golmankhaneh, K.A., Golmankhaneh, K.A., Nigmatullin, R.R.: Newtonian law with memory. Nonlinear Dyn. (2009). doi:10.1007/s11071-009-9581-1
10. Baleanu, D., Golmankhaneh, K.A., Nigmatullin, R.R., Golmankhaneh, K.A.: Fractional Newtonian mechanics. Cent. Eur. J. Phys. (2009). doi:10.2478/s11534-009-0085-x
11. Baleanu, D., Golmankhaneh, A.K., Golmankhaneh, A.K.: Fractional Nambu mechanics. Int. J. Theor. Phys. **48**, 1044 (2009)
12. Faddeev, L., Jackiw, R.: Hamiltonian reduction of unconstrained and constrained systems. Phys. Rev. Lett. **60**, 1692–1694 (1988)
13. Golmankhaneh, K.A.: Relativistic scalar fields for non-conservative system. Phys. Scr. (2009). doi:10.1088/0031-8949/2009/T136/014008
14. Gorenflo, R., Mainardi, F.: Fractional Calculus: Integral and Differential Equations of Fractional Orders, Fractals and Fractional Calculus in Continuum Mechanics. Springer, New York (1997)
15. Gastao, S.F.F., Delfim, F.M.T.: A formulation of Noether's theorem for fractional problems of calculus of variations. J. Math. Anal. Appl. **334**, 834–846 (2007)
16. Henneaux, M., Teitelboim, C.: Quantization of Gauge Systems. Princeton University Press, Princeton (1992)

17. Hilfer, R.: Application of Fractional Calculus in Physics. World Scientific, Singapore (2000)
18. Kilbas, A.A., Srivastava, H.H., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
19. Kishore, B.: Lecture Notes in Physics. Springer, Berlin (1983)
20. Klimek, K.: Fractional sequential mechanics-models with symmetric fractional derivative. Czech. J. Phys. **51**, 1348–1354 (2001)
21. Klimek, K.: Lagrangean and Hamiltonian fractional sequential mechanics. Czech. J. Phys. **52**, 1247–1253 (2002)
22. Kulshreshtha, D.S., Müller-Kirsten, H.J.W.: Faddeev-Jackiw quantization of self-dual fields. Phys. Rev. D **45**, 393–397 (1992)
23. Laskin, N.: Fractional quantum mechanics. Phys. Rev. E **62**, 3135–3145 (2000)
24. Miller, K.S., Ross, B.: An Introduction to the Fractional Integrals and Derivatives-Theory and Application. Wiley, New York (1993)
25. Muslih, S.I., Baleanu, D.: Formulation of Hamiltonian Equations for Fractional Variational Problems. Czech. J. Phys. **55**, 633–642 (2005)
26. Muslih, S.I., Baleanu, D.: Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives. J. Math. Anal. Appl. **304**, 599–606 (2005)
27. Muslih, S.I., El-Zalan, H.A., Rabei, E.M.: Hamilton-Jacobi quantization of singular Lagrangians with linear velocities. Int. J. Theor. Phys. **44**, 1271–1279 (2005)
28. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
29. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
30. Riewe, F.: Nonconservative Lagrangian and Hamiltonian mechanics. Phys. Rev. E **53**, 1890–1899 (1996)
31. Riewe, F.: Mechanics with fractional derivatives. Phys. Rev. E **55**, 3581–3592 (1997)
32. Rabei, E.M., Nawafleh, K.I., Hiijawi, R.S., Muslih, S.I., Baleanu, D.: The Hamilton formalism with fractional derivatives. J. Math. Anal. Appl. **327**, 891–897 (2007)
33. Rabei, E.M., Almayteh, I., Muslih, S.I., Baleanu, D.: Hamilton-Jacobi formulation of systems with Caputo's fractional derivative. Phys. Scr. **77**, 015101 (2007)
34. Rabei, E., Tarawneh, D.M., Muslih, S.I., Baleanu, D.: Heisenberg's equations of motion with fractional derivatives. J. Vibr. Control **13**, 239–247 (2007)
35. Rabei, E.M., Altarazi, I.M.A., Muslih, S.I., Baleanu, D.: Fractional WKB approximation. Nonlinear Dyn. **57**(1–2), 171–175 (2009)
36. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives Theory and Applications. Gordon and Breach, New York (1993)
37. Sundermeyer, K.: Constrained Dynamics. Lecture Notes in Physics, vol. 169. Springer, New York (1982)
38. Tarasov, V.E.: Fractional variation for dynamical systems: Hamilton and Lagrange approaches. J. Phys. **39**(26), 8409–8425 (2006)
39. Tarasov, V.E., Zaslavsky, G.M.: Fractional Ginzburg-Landau equation for fractal media. Physica A **354** (2005)
40. West, B.J., Bologna, M., Grigolini, P.: Physics of Fractal Operators. Springer, New York (2003)
41. Zaslavsky, G.M.: Chaos, fractional kinetics, and anomalous transport. Phys. Rep. **371**, 461–580 (2002)
42. Zaslavsky, G.M.: Hamiltonian Chaos and Fractional Dynamics. Oxford University Press, Oxford (2005)